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Q-instantons

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Abstract: We construct the half-supersymmetric instanton solutions that are electric-magnetically dual to the recently discussed half-supersymmetric Q7-branes. We call these instantons "Q-instantons". Whereas the D-instanton is most conveniently described using the RR axion χ and the dilaton ϕ, the Q-instanton is most conveniently described using a different set of fields (χ', T), where χ' is an axionic scalar. The real part of the Q-instanton on-shell action is a function of T and the imaginary part is linear in χ'. Discrete shifts of the axion χ' correspond to $\mathrm{PSL}(2,\mathbb{Z})$ transformations that are of finite order. These are e.g. pure S-duality transformations relating weak and strongly coupled regimes. We argue that near each orbifold point of the quantum axion-dilaton moduli space $\left\{\tau \mid \tau \in \frac{\mathrm{PSL}(2,\mathbb{R})}{\mathrm{SO}(2)\times\mathrm{PSL}(2,\mathbb{Z})}\right\}$ the higher order $\mathcal{R}^4$ terms in the string effective action contain contributions from an infinite sum of single multiply-charged instantons with the Q-instantons corresponding to the orbifold points $\tau = i, \rho$.

Keywords: Solitons Monopoles and Instantons, p-branes, Gauge Symmetry.

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Contents

1. Introduction

Recently, 7-brane configurations have been investigated with an emphasis on their supersymmetry properties [1] and their coupling to the bulk IIB supergravity fields [2]. As shown in [1] generic 7-brane configurations contain 7-branes that are associated to various $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes. The $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes have been classified in [3] and [4]. In the classical theory, i.e. if one does not take into account charge quantization there are three families of $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes depending on whether $\det Q < 0$, $\det Q = 0$ or $\det Q > 0$ where Q is such that $e^Q \in \mathrm{SL}(2,\mathbb{R})$. The (p,q) 7-branes correspond to the case $\det Q = 0$ whereas the Q7-branes have $\det Q > 0$ (no 7-branes correspond to $\det Q < 0$). For each of these 7-branes one can define an axion, that we denote by χ', with respect to which the 7-brane is magnetically charged. When we consider charge quantization the $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes with $\det Q = 0$ are given by $\pm T^n$ (with $n = 0,1,2,\ldots$), while for $\det Q > 0$ they are given by S, $-S$, $(T^{-1}S)^{\pm 1}$ and $(-T^{-1}S)^{\pm 1}$, where T and S transform the axion-dilaton $\tau = \chi + ie^{-\phi}$ as $T\tau = \tau + 1$ and $S\tau = -1/\tau$, respectively.

$\mathrm{SL}(2,\mathbb{Z})$ conj. class.	branes
T^n	n D7-branes
$(-S)^n\,,\, n \leq 4$	n $(-S)$-branes
$(-T^{-1}S)^n\,,\, n \leq 6$	n $(-T^{-1}S)$-branes

Table 1: $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes and branes; $n \in \mathbb{N}$.

In the notation of [1, 2] the $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes $-S$ and $-T^{-1}S$ correspond to a single positive tension Q7-brane. In table 1 we give an interpretation of the $\det Q = 0$ and $\det Q > 0$ $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes in terms of D7-branes as well as in terms of positive tension Q7-branes.

The objects that are electrically charged under χ' and that are dual to a positive tension Q7-brane are dubbed "Q-instantons". These are half-BPS solutions of Euclidean IIB supergravity. It is well-known that the object that is dual to the D7-brane is the so-called D-instanton [5]. In this paper we present a path integral analysis of the Q-instantons providing us with their tunneling interpretation and we derive their basic physical properties such as their charge and on-shell Euclidean action.

At first sight the existence of new half-supersymmetric instanton solutions to Euclidean IIB supergravity might be surprising since a simple analysis of the Killing spinor equations and field equations seems to lead to the unique D-instanton solution of [5], up to an $\mathrm{SL}(2,\mathbb{Z})$ transformation of the D-instanton into a (p,q)-instanton. There remains however the possibility that there exist instantonic solutions that differ from the D-instanton due to a difference in the source and boundary term. This is in fact what happens and it leads to different on-shell actions for the D- and Q-instantons.

In [6] it was shown that the D-instanton contributes to higher order corrections to the string effective action of the form of $\mathcal{R}^4$ terms. Since Q-instantons preserve the same supersymmetries as the D-instanton, they are expected to contribute to the same $\mathcal{R}^4$ terms as well. We will argue that this is the case.

The paper is organized as follows. In section 2 we first discuss the general idea of a Q-brane in type IIB supergravity. In section 3 we review the construction and properties of the D-instanton. These results are compared in section 4 with the Q-instanton source and boundary terms and its on-shell action. In section 5 we make a path integral analysis of the Q-instanton and in section 6 we discuss the Q-instanton contribution to the $\mathcal{R}^4$ terms. We end with a discussion of our results in section 7.

2. Q-branes

Before discussing the Q-instantons we first outline some general ideas regarding the concept of Q-branes.

The low energy description of the type IIB superstring in a bosonic background with vanishing 3-form and 5-form field strengths is described by the well-known axion-dilaton

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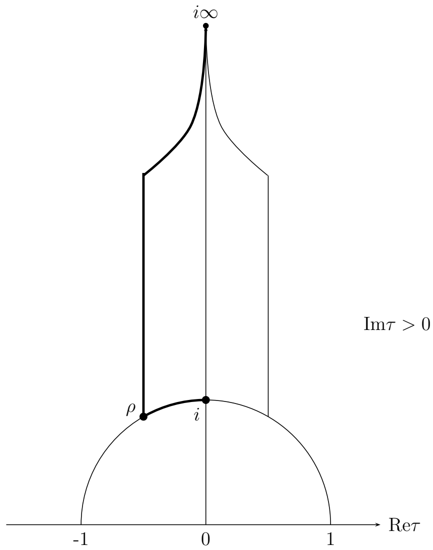

Figure 1: Fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})\backslash \mathrm{PSL}(2,\mathbb{R})/\mathrm{SO}(2)$

action coupled to gravity

$$S = \int_{\mathcal{M}_{9,1}} \left(\star 1 R - \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) , \tag{2.1}$$

where the vacuum expectation value of e^{ϕ} gives the string coupling g_S. The quantum moduli space of the inequivalent values of the complex axion-dilaton field $\tau = \chi + ie^{-\phi}$ of the non-perturbative type IIB string theory is conjectured [7] to be given by the orbifold

$$\frac{\mathrm{PSL}(2,\mathbb{R})}{\mathrm{SO}(2) \times \mathrm{PSL}(2,\mathbb{Z})} . \tag{2.2}$$

This orbifold is depicted in figure 1 in which we have exaggerated the cusp-like behavior near the orbifold point $\tau_0 = i\infty$ where the string coupling g_S is close to zero. It is this region of the moduli space where we know that there is agreement between results from the perturbatively defined type IIB superstring theory and IIB supergravity. For example, the spectrum of the D-branes predicted by perturbative IIB string theory corresponds to brane-like solutions of IIB supergravity [8]. By ρ we denote the point $\rho = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

To indicate the place of the new Q7-branes in the fundamental domain of the axion-dilaton moduli space, let us recall that the conjectured $\mathrm{SL}(2,\mathbb{Z})$ duality of IIB superstring theory leads to the notion of (p,q) branes, i.e. extended objects on which a (p,q) string ends. The numbers p and q are two non-negative integers. From the point of view of the fundamental string, the F1-string or the $(1,0)$ string whose coupling constant goes to zero in the (perturbative) region of the moduli space at the point $\tau_0 = i\infty$, the (p,q) string is a bound state of p F1-strings and q D1-branes [9]. Alternatively, one can consider a (p,q) string with p and q relatively prime as an elementary string whose perturbative sector is determined by the zero limit of the coupling constant associated with $e^{\phi_{q,r}}$ [10],

$$e^{\phi_{q,r}} \equiv q e^{\phi} \left|\tau + \frac{r}{2q}\right|^2 \quad \text{with} \quad r = 2\sqrt{pq}\,, \tag{2.3}$$

in which case the relevant part of the moduli space is a cusp around the point $\tau_0 = -r/2q$ (where $e^{\phi_{q,r}} \to 0$). The low energy description of the (p,q) string theory is again described by the action (2.1). This can be made manifest by writing eq. (2.1) in the following form

$$S = \int_{\mathcal{M}_{9,1}} \left(\star 1 R - \frac{1}{2} \star d\phi_{q,r} \wedge d\phi_{q,r} - \frac{1}{2} e^{2\phi_{q,r}} \star d\chi_{q,r} \wedge d\chi_{q,r}\right) . \tag{2.4}$$

The form of the action (2.4) suggests that each (p,q) string vacuum, i.e. the point where $e^{\phi_{q,r}} \to 0$, has its own coupling constant $< e^{\phi_{q,r}} >$ and axion $\chi_{q,r}$. The passage from one (p,q) string vacuum to another and corresponding field redefinitions are, of course, governed by the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the type IIB string theory.

Let us now introduce, using the notation of [2], the $\mathrm{SL}(2,\mathbb{R})$ algebra valued charge matrix

$$Q = \begin{pmatrix} r/2 & p \\ -q & -r/2 \end{pmatrix} , \tag{2.5}$$

which describes the electric coupling of the Q7-brane to an $\mathrm{SL}(2,\mathbb{R})$ triplet of 8-forms (see eqs. (2.15) and (2.18) below). The conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ are characterized by the value of the trace of e^Q,

$$e^Q = \cos(\sqrt{\det Q})\mathbb{1} + \frac{\sin(\sqrt{\det Q})}{\sqrt{\det Q}} Q\,. \tag{2.6}$$

Families of $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes are formed by

$$\mathrm{tr}\, e^Q = 2\cos(\sqrt{\det Q}) \begin{cases} = 2 \\ > 2 \\ < 2 \end{cases} \quad \text{or equivalently, by} \quad \det Q \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases} . \tag{2.7}$$

When we add D-branes to the type IIB supergravity theory the duality group $\mathrm{SL}(2,\mathbb{R})$ is broken down to the subgroup that is generated by the shift symmetry of the RR axion, i.e. the $\mathbb{R}$ subgroup of $\mathrm{SL}(2,\mathbb{R})$. This for example implies that all the D-brane actions are invariant under the shift of the RR axion. Likewise, when we add a Q-brane to the type IIB supergravity theory the duality group $\mathrm{SL}(2,\mathbb{R})$ is broken down to the subgroup that is generated by the shift symmetry of the χ', i.e. the $\mathrm{SO}(2)$ subgroup of $\mathrm{SL}(2,\mathbb{R})$. Hence, all the brane solutions of IIB supergravity are associated to fixed points of e^Q with either $\det Q = 0$ or $\det Q > 0$. The case $\det Q = 0$ corresponds to the (p,q) branes[1] and the case $\det Q > 0$ corresponds to Q-branes. The case $\det Q < 0$ does not arise because there are

[1]The $\mathrm{SL}(2,R)$ covariant actions which describe the (p,q)-branes have been constructed in [11, 12]. They can be regarded as the $\mathrm{SL}(2,R)$ transformed Dp-brane actions.

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no fixed points of e^Q with $\det Q < 0$ that are part of the quantum moduli space (2.2). The point τ_0 is a fixed point under the e^Q transformation if it satisfies the equation

$$e^Q \tau_0 = \frac{a\tau_0 + b}{c\tau_0 + d} = \tau_0 \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^Q \,. \tag{2.8}$$

The fixed points τ_0 of e^Q with $q > 0$ and $0 \leq \operatorname{Im}\tau_0 < \infty$ for $\det Q \geq 0$ are given by

$$\tau_0 = -\frac{r}{2q} + \frac{i}{q}\sqrt{\det \mathrm{Q}} \,. \tag{2.9}$$

As shown in [2] the Q7-brane configurations are most conveniently described in terms of the variables T and χ' which (for $q > 0$ [2]) are defined by the following relations

$$\frac{\tau - \tau_0}{\tau - \bar{\tau}_0} = e^{2i\sqrt{\det Q}\,\mathcal{T}} \,, \tag{2.10}$$

where $\mathcal{T}$ is given by

$$\mathcal{T} = \chi' + \frac{i}{4\sqrt{\det \mathrm{Q}}} \log \frac{T + 2\sqrt{\det \mathrm{Q}}}{T - 2\sqrt{\det \mathrm{Q}}} \,, \qquad \chi' \sim \chi' + \frac{\pi}{\sqrt{\det Q}} \,. \tag{2.11}$$

As follows from eq. (2.10) the values χ' and $\chi' + \pi/\sqrt{\det Q}$ are to be identified. In the limit $\det Q \to 0$ eq. (2.11) reduces to

$$\mathcal{T} \to \frac{-1}{q\left(\tau + \frac{r}{2q}\right)} \quad \text{with} \quad r = 2\sqrt{pq} \,. \tag{2.12}$$

The requirement that $\operatorname{Im}\tau > 0$ implies that $T > 2\sqrt{\det Q}$ or what is the same $\operatorname{Im}\mathcal{T} > 0$. The relation (2.10) between τ and $\mathcal{T}$ is a conformal mapping from the upper half plane $\operatorname{Im}\tau > 0$ to the vertical strip

$$\left\{ \mathcal{T} \mid \operatorname{Im}\mathcal{T} > 0 \quad \text{and} \quad \chi' \sim \chi' + \frac{\pi}{\sqrt{\det Q}} \right\} \,. \tag{2.13}$$

For special values of p, q, and r the point τ_0 is equal to the points i and ρ of figure 1. In terms of $\mathcal{T}$ the region close to i or ρ looks like a cusp, that is, a region where $\operatorname{Im}\mathcal{T} \to \infty$ while χ' becomes undetermined as τ approaches τ_0.

Table 2 shows the values of p, q, r for each of the orbifold points τ_0 of Fig 1 and the related $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes. The periodicity of the axion χ' is determined by the value of $\pi/\sqrt{\det Q}$. The discrete isometries are the $\mathrm{PSL}(2,\mathbb{Z})$ transformations generated by T and S with $T\tau = \tau + 1$ and $S\tau = -1/\tau$. The periodicity of χ' follows from the fact that the transformations S and $T^{-1}S$ are, respectively, of order 2 and 3 in $\mathrm{PSL}(2,\mathbb{Z})$.

In terms of the fields T and χ' the action (2.1) takes the form

$$S = \int_{\mathcal{M}_{9,1}} \left(\star 1 R - \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT - \frac{1}{2}(T^2 - 4\det Q) \star d\chi' \wedge d\chi' \right) \,. \tag{2.14}$$

[2]The restriction $q > 0$ guarantees that T, which is the tension of a Q7-brane, is postive [2].

τ_0	(p,q,r)	$\pi/\sqrt{\det Q}$	discrete isometry	$\mathrm{SL}(2,\mathbb{Z})$ conj. class
$i\infty$	$(1,0,0)$	∞	$T:\chi\to\chi+1$	T
i	$\left(\frac{\pi}{2},\frac{\pi}{2},0\right)$	2	$-S:\chi'\to\chi'+1$	$-S$
ρ	$\left(\frac{2\pi}{3\sqrt{3}},\frac{2\pi}{3\sqrt{3}},\frac{2\pi}{3\sqrt{3}}\right)$	3	$-T^{-1}S:\chi'\to\chi'+1$	$-T^{-1}S$

Table 2: Properties of the orbifold points $\tau_0 = i\infty, i, \rho$.

This form of the IIB supergravity action is obtained using the field redefinition (2.10). The dependence of eq. (2.14) on the parameter $\det Q$ can be removed by making the inverse field redefinition. However, when we couple the action (2.14) to the Q7-brane action the dependence on the conjugacy class parameter $\det Q$ cannot be eliminated since there is no field redefinition which takes (2.14) coupled to a Q7-brane to (2.1) coupled to a D7-brane [2]. The same difference we shall observe in the case of D- and Q-instantons. This is to be contrasted to the case of IIB supergravity coupled to (p,q)-branes in which case there always exists a field redefinition that transforms the system of IIB supergravity coupled to a (p,q)-brane to IIB supergravity coupled to the corresponding D-brane.

Following [2] we introduce a 9-form field strength which is dual to $d\chi'$

$$(T^2 - 4\mathrm{det}Q)\, d\chi' = \star(pF_9 + qH_9 + rG_9) \equiv \star\mathcal{F}_9\,, \tag{2.15}$$

where the 9-forms F_9, H_9, G_9 are organized in a triplet transforming in the adjoint of $\mathrm{SL}(2,\mathbb{R})$ and p, q and r are the components of the matrix Q (2.5). From the axion χ' equation of motion (when ignoring its coupling to the 2-forms and 6-forms) it follows that

$$d\mathcal{F}_9 = 0\,, \tag{2.16}$$

so that locally

$$\mathcal{F}_9 = d\mathcal{A}_8. \tag{2.17}$$

The Q7-brane minimally couples[3] to $\mathcal{A}_8$ via the Wess-Zumino term

$$S^{Q7}_{\min} = m\int \mathcal{A}_8\,, \tag{2.18}$$

where m is the Q7-brane electric charge with respect to $\mathcal{A}_8$, or magnetic charge associated with its axion dual χ'. In [5] it has been shown that the axion charge m of the (p,q) 7-branes takes discrete values ($m = 1, 2, 3, \ldots$) while the discreteness of m for the Q7-branes was shown in [2].

When $p = 1$ and $q = r = 0$ eq. (2.15) describes the duality between the RR axion χ and the RR 8-form C_8 whose field strength is F_9. When $q = 1$ and $p = r = 0$ the field strength is $H_9 = dB_8$ where B_8 is an NSNS 8-form that couples to the NSNS 7-brane (the S-dual transformed D7-brane). The case $p = q = 0$ and $r \neq 0$ corresponds to $\det Q < 0$ and hence there is no 7-brane that couples only to a D_8 field whose field strength is $G_9 = dD_8$.

[3] For an early discussion on instantons and monopole-like configurations related to $(d-2)$-form gauge fields we refer to [13].

The dynamics of the 9-form $\mathcal{F}_9$ can be described by the following first order action

$$S\left[g_{\mu\nu}, \chi', \mathcal{F}_9, T\right] = \int_{\mathcal{M}_{9,1}} \Bigg(*1R - \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT \\ - \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 - \chi' d\mathcal{F}_9 \Bigg)\,. \tag{2.19}$$

In the action (2.19) the axion χ' appears (in a shift symmetry invariant way) as a Lagrange multiplier. The term $d\mathcal{F}_9$ is parity odd as is χ'.[4] The variation of (2.19) with respect to $\mathcal{F}_9$ gives the duality relation (2.15). If we substitute this relation back into the action we obtain the action (2.14). If we vary (2.19) with respect to χ' we find the Bianchi identity for $\mathcal{F}_9$ (2.16). If we substitute its solution (2.17) back into the action (2.19) we obtain a second order action for $\mathcal{A}_8$. The action (2.19) will be the starting point of our discussion of the Q-instantons.

3. D-instantons

Before discussing the new Q-instanton solutions of IIB supergravity, let us briefly review the derivation of the D-instanton solution [5]. This is a solution of the equations of motion of the axion and dilaton coupled to gravity in Euclidean space. The Wick rotation of the action (2.1) is carried out by taking into account that the axion is an axial scalar and hence gets replaced with $i\chi$.[5] The Wick rotation thus changes the sign of the Einstein term and the dilaton kinetic term leaving intact the sign of the axion kinetic term. So the Euclidean action is

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right)\,. \tag{3.1}$$

Note that the action is invariant under the axion shift symmetry $\chi \to \chi + b$ with b being a constant real parameter. The Einstein equations and the equations of motion of the axion and the dilaton, which follow from (3.1), have the form

$$R_{mn} - \frac{1}{2}\left(\partial_m\phi\partial_n\phi - e^{2\phi}\partial_m\chi\partial_n\chi\right) = 0\,, \tag{3.2}$$

$$D_m(e^{2\phi}D^m\chi) = 0\,, \tag{3.3}$$

$$D_m D^m\,\phi + e^{2\phi}\,(\partial\chi)^2 = 0\,. \tag{3.4}$$

The Ansatz imposed on the fields to get the D-instanton solution of (3.2)–(3.4) is

$$g_{mn} = \delta_{mn}, \quad d\chi = \pm e^{-\phi}\, d\phi = \mp d e^{-\phi}. \tag{3.5}$$

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[4]The parity oddness of χ' can be understood as follows. The field redefinition (2.10) and (2.11) implies that we have

$$-e^{\phi}(\chi - \mathrm{Re}\,\tau_0) = \frac{(T^2 - 4\det Q)^{1/2}}{2\sqrt{\det Q}}\,\sin 2\sqrt{\det Q}\,\chi'\,. \tag{2.20}$$

Then it follows from the relation (2.20) that χ' has the same parity as χ. Since, the RR axion is parity odd so is χ'.

[5]We anticipate that in section 5 it will be shown that from the path integral point of view it is not allowed to send χ to $i\chi$ (or χ' to $i\chi'$ when it concerns the Q-instanton) under a Wick rotation. Since, in this and the next section we discuss classical Euclidean field theory which only provides on-shell information about the saddle point approximation there is no harm done in sending χ to $i\chi$.

The equation (3.5) is nothing but the Bogomol'nyi bound saturation condition imposed on the axion-dilaton system in flat space. The upper and lower signs in (3.5) correspond, respectively to the D-instanton and anti-D-instanton. When (3.5) is imposed eqs. (3.2)–(3.4) reduce to

$$\partial_m(e^{2\phi}\partial^m\chi) = 0 \qquad \rightarrow \qquad \partial^2 e^\phi = 0\,, \tag{3.6}$$

$$\partial_m\partial^m\phi + (\partial\phi)^2 = 0 \qquad \rightarrow \qquad e^{-\phi}\,\partial^2 e^\phi = 0. \tag{3.7}$$

A spherically symmetric solution to the above equations which describes a single (anti-)instanton is

$$e^\phi = e^{\phi_\infty} + \frac{c}{r^8}, \quad \chi - \chi_\infty = \mp\left(e^{-\phi} - e^{\phi_\infty}\right), \tag{3.8}$$

where the upper sign stands for the instanton and the lower sign corresponds to the anti-instanton, ϕ_∞ and χ_∞ are the values of the dilaton and axion at $r = \sqrt{x^m x_m} = \infty$ and $c > 0$ is (roughly speaking) the instanton charge, namely,

$$c = \frac{2\pi|n|}{8\mathrm{Vol}(S^9)}\,, \tag{3.9}$$

with $\mathrm{Vol}(S^9)$ being the volume of a 9-sphere of a unit radius and n being an integer which manifests the instanton charge quantization [5]. Note that from (3.8) it follows that for the instanton $\chi + e^{-\phi}$ is constant and for the anti-instanton $\chi - e^{-\phi}$ is constant everywhere in 10d space.

The solution (3.8) is singular at $r = 0$ which implies that it is sourced by a point-like object (the instanton) sitting at $r = 0$. The (anti-)instanton contribution to the right hand side of the axion-dilaton field equations (3.6) and (3.7) is as follows

$$\partial_m(e^{2\phi}\partial^m\chi) = \mp 2\pi|n|\,\delta^{(10)}(\vec{x})\,, \qquad e^{-\phi}\,\partial^2 e^\phi = -2\pi|n|\,e^{-\phi}\,\delta^{(10)}(\vec{x})\,. \tag{3.10}$$

Eqs. (3.10) can be obtained by varying the supergravity action (3.1) coupled to the instanton source

$$S = \int_{\mathcal{M}_{10}}\left(-\star 1R + \frac{1}{2}\star d\phi\wedge d\phi - \frac{1}{2}e^{2\phi}\star d\chi\wedge d\chi\right) + 2\pi|n|\int_{\mathcal{M}_{10}}\delta^{(10)}(\vec{x})\left(e^{-\phi}\pm\chi\right)\star 1\,, \tag{3.11}$$

and imposing the Ansatz (3.5).

The presence of the instanton source term breaks the invariance of the action (3.11) under the shift symmetry $\chi \rightarrow \chi + b$. The invariance can be restored by adding to eq. (3.11) the boundary term

$$-\int_{\partial\mathcal{M}_{10}}\chi\, e^{2\phi}\star d\chi = -\int_{\mathcal{M}_{10}} d(\chi\, e^{2\phi}\star d\chi) = \int_{\mathcal{M}_{10}} d^{10}x\,\partial_m(\chi\, e^{2\phi}\,\partial^m\chi)\,, \tag{3.12}$$

such that

$$\int_{\partial\mathcal{M}_{10}} e^{2\phi}\star d\chi = \pm\,2\pi|n|\int_{\mathcal{M}_{10}}\delta^{(10)}(\vec{x})\star 1 = \pm\,2\pi|n|\,. \tag{3.13}$$

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Note that this boundary condition is compatible with eqs. (3.10).

The appearance of the boundary term (3.12) in the supergravity action can be best understood if one starts from the action which includes the field strength $F_9 = dA_8$ of the 8-form gauge field A_8 and then dualizes it into the axion action by adding the term $\int_{\mathcal{M}_{10}} \chi dF_9$ (compare with (2.19))

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1R + \frac{1}{2} \star d\phi \wedge d\phi + \frac{1}{2} e^{-2\phi} \star F_9 \wedge F_9 \right) + \int_{\mathcal{M}_{10}} \chi \, dF_9 \,. \tag{3.14}$$

If in (3.14) the field F_9 is considered as the independent one (i.e. not a curl of A_8), the variation with respect to this field gives the duality relation $F_9 = e^{2\phi} \star d\chi$ which can be substituted back into the action (3.14) thus reducing it to

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) - \int_{\partial\mathcal{M}_{10}} \chi \, e^{2\phi} \star d\chi \,. \tag{3.15}$$

The boundary term we have looked for has appeared as a result of the integration by parts of the last term in (3.14). To summarize, the shift symmetry invariant action for the IIB supergravity - D-instanton system is

$$\begin{aligned} S = & \int_{\mathcal{M}_{10}} \left(- \star 1R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) - \int_{\partial\mathcal{M}_{10}} \chi \, e^{2\phi} \star d\chi \\ & + 2\pi |n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(e^{-\phi} \pm \chi \right) \star 1 \,. \end{aligned} \tag{3.16}$$

We are now ready to compute the on-shell value of this action by substituting into (3.16) the instanton solution (3.5), (3.8). Then the bulk part of the action vanishes because of the Bogomol'nyi bound saturation, the contribution from the boundary term gets canceled by the χ part of the source term, and we are left with

$$S_D|_{\text{on-shell}} = 2\pi |n| \, e^{-\phi_\infty} , \tag{3.17}$$

where e^{ϕ_∞} is the string coupling constant.

In section 5 it will be shown that the result (3.17) corresponds to a saddle point approximation of a path integral that computes the transition amplitude between axion conjugate momentum eigenstates or, what is the same, between Noether charge eigenstates of the Noether current, associated to the shift symmetry $\chi \to \chi + b$, that differ by n units. As shown in section 5 (see also [6]), in order to obtain a saddle point approximation between axion χ eigenstates, one must add to (3.17) the imaginary term

$$-2\pi n i \chi_\infty \,, \tag{3.18}$$

with $n > 0$ for the D-instanton and $n < 0$ for the anti-D-instanton. The axion that appears in (3.18) is the RR axion χ of the Lorentzian IIB theory (and not the Wick rotated one of this section). Thus the D-instanton action takes the form

$$S_D = -2\pi i |n| \tau_\infty \,. \tag{3.19}$$

4. Q-instantons

Let us now perform an analysis similar to the one described above to find instanton solutions of IIB supergravity for which $\det Q > 0$ using the fields (T, χ').

4.1 Q-instanton action

The analog of eq. (3.14), that should provide us with the relevant boundary term in the Q-instanton action, is the Euclidean version of the action (2.19), namely

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \frac{1}{T^2 - 4\det Q} \star dT \wedge dT + \frac{1}{2} \frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 + \chi' d\mathcal{F}_9 \right) , \tag{4.1}$$

where we replaced χ' by $i\chi'$. The $\mathcal{F}_9$ equation of motion gives the duality relation between $\mathcal{F}_9$ and the Wick rotated χ' (similar to (2.15)). Substituting the duality relation back into the action we get

$$\begin{aligned} S = &\int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \frac{1}{T^2 - 4\det Q} \star dT \wedge dT - \frac{1}{2} (T^2 - 4\det Q) \star d\chi' \wedge d\chi' \right) \\ &- \int_{\partial \mathcal{M}_{10}} (T^2 - 4\det Q)\, \chi' \star d\chi' \, . \end{aligned} \tag{4.2}$$

To this action we should couple an instanton source term that will be the counterpart of (3.16) in the (T, χ') basis. Remember that the form of the D-instanton coupling term was prompted by the structure of the source terms on the right-hand side of the axion-dilaton eqs. (3.10) which take care of the singularity of the D-instanton solution (3.8). So to find the relevant form of the instanton coupling term in the new basis we should study the (T, χ') equations of motion.

As in the D-instanton case we shall assume that for the solution under consideration the $d = 10$ space is flat and, as follows from the Einstein equation, T and χ' are related by the following Bogomol'nyi bound saturation condition

$$d\chi' = \pm (T^2 - 4\det Q)^{-1}\, dT \, . \tag{4.3}$$

Then the χ'– and T-field equations that follow from (4.2) and that satisfy (4.3) take, respectively, the following form

$$- \partial_m ((T^2 - 4\det Q)\, \partial^m \chi') = 0, \tag{4.4}$$

$$(T^2 - 4\det Q)^{-1}\, \partial_m \partial^m T = \frac{1}{4\sqrt{\det Q}} \left(\log \frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \right)' \partial_m \partial^m T = 0 \, , \tag{4.5}$$

where the prime over the logarithm denotes its derivative with respect to T. From the form of (4.4) and (4.5) we see that for these equations to acquire the delta-function source terms $2\pi |n|\, \delta^{(10)}(\vec{x})$ we should add to the action (4.2) a coupling term of the form[6]

$$2\pi |n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(\frac{1}{4\sqrt{\det Q}} \log \frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \pm \chi' \right) \star 1 \, . \tag{4.6}$$

[6]The source term (4.6) is uniquely specified by requiring an electric coupling term linear in χ' and by requiring it to preserve the same supersymmetries as the D-instanton.

The source term guarantees that the instanton solution we are interested in is defined on the entire 10d Euclidean space. As in the D-instanton case, the Q-instanton charge is quantized ($|n| = 1, 2, 3, \ldots$) as we shall demonstrate in the next Subsection.

Upon adding (4.6) to eq. (4.2) we get the following action in flat Euclidean space

$$\begin{aligned} S = & \int_{\mathcal{M}_{10}} \left(\frac{1}{2} \frac{1}{T^2 - 4\det Q} dT \wedge \star dT - \frac{1}{2}(T^2 - 4\det Q) d\chi' \wedge \star d\chi' \right) \\ & - \int_{\partial\mathcal{M}_{10}} (T^2 - 4\det Q)\, \chi' \star d\chi' \\ & + 2\pi|n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(\frac{1}{4\sqrt{\det Q}} \log \frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \pm \chi' \right) \star 1\,. \end{aligned} \tag{4.7}$$

As in the D-instanton case, due to the presence of the source term there is only one boundary, $\partial\mathcal{M}_{10}$, which is located at $r = \sqrt{x^m x_m} = \infty$. The action (4.2) is invariant under arbitrary shifts of the axion, $\chi' \to \chi' + b$ (where b is any real number) provided that

$$\int_{\partial\mathcal{M}_{10}} (T^2 - 4\det Q) \star d\chi' = \pm\, 2\pi|n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \,\star 1 \,= \pm\, 2\pi|n|. \tag{4.8}$$

The equations of motion of the fields χ' and T which follow from eq. (4.7) acquire the contribution of the instanton source term and take the following form on the Bogomol'nyi bound (4.3)

$$\partial_m((T^2 - 4\det Q)\, \partial^m \chi') \,=\, \mp\, 2\pi|n|\delta^{(10)}(\vec{x})\,, \tag{4.9}$$

$$\partial_m\partial^m T \,=\, -2\pi|n|\delta^{(10)}(\vec{x})\,. \tag{4.10}$$

The Q-instanton solution to eqs. (4.3) and (4.10) is

$$\chi' - \chi'_\infty = \mp \left[\frac{1}{4\sqrt{\det Q}} \log\left(\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \right) - \frac{1}{4\sqrt{\det Q}} \log\left(\frac{T_\infty + 2\sqrt{\det Q}}{T_\infty - 2\sqrt{\det Q}} \right) \right], \tag{4.11}$$

$$T = T_\infty + \frac{c}{r^8}\,, \tag{4.12}$$

where c has been given in eq. (3.9). Substituting this solution into the action (4.7) we get the on-shell value of the Q-instanton action

$$S_Q\,|_{\text{on-shell}} = \frac{\pi|n|}{2\sqrt{\det \mathrm{Q}}} \log \frac{T_\infty + 2\sqrt{\det \mathrm{Q}}}{T_\infty - 2\sqrt{\det \mathrm{Q}}}\,. \tag{4.13}$$

By virtue of the definition of T, eqs. (2.10) and (2.11), when $\det \mathrm{Q} \to 0$ the action (4.13) reduces to the (p, q)-instanton action where p and q are as in eq. (2.12).

In section 5 it will be shown that the result (4.13) corresponds to a saddle point approximation of a path integral that computes the transition amplitude between Noether charge eigenstates of the Noether current, associated to the shift symmetry $\chi' \to \chi' + b$, that differ by n units. As will also be shown in section 5, in order to obtain a saddle point approximation between axion χ' eigenstates one must add to (3.17) the imaginary term

$$-2\pi n i \chi'_\infty\,, \tag{4.14}$$

with $n > 0$ for a Q-instanton and $n < 0$ for an anti-Q-instanton. The axion that appears in (4.14) is the axion χ' of the Lorentzian IIB theory (and not the Wick rotated one of this section). The Q-instanton action thus acquires the form

$$S_Q = -2\pi i|n|\mathcal{T}_\infty \,. \tag{4.15}$$

It is instructive to compare the actions (4.7) and (4.13) and the equations of motion (4.9) and (4.10) with the D-instanton case of the previous section. This will allow us to understand at which point the Q-instanton solutions to (4.7) are different from the D-instanton solutions. To this end let us perform the following field redefinition[7] whose form is prompted by the fact that the Bogomol'nyi bounds (3.5) and (4.3) must be field redefinition equivalent as both follow from the Einstein equation with $g_{\mu\nu} = \delta_{\mu\nu}$. The field redefinition relating (3.5) and (4.3) is

$$T = 2\sqrt{\det Q}\, e^{\phi}\chi\,, \qquad \chi' = \frac{1}{4\sqrt{\det Q}} \log\left(\chi^2 - e^{-2\phi}\right)\,. \tag{4.16}$$

Using (4.16) the action (4.2) in flat space takes the form

$$\begin{aligned} S = &\int_{\mathcal{M}_{10}} \left(\frac{1}{2}\star d\phi\wedge d\phi - \frac{1}{2}e^{2\phi}\star d\chi\wedge d\chi\right) \\ &-\frac{1}{2}\int_{\partial\mathcal{M}_{10}} \log\left(\chi^2 - e^{-2\phi}\right)\,\star\,(e^{2\phi}\,\chi\, d\chi + d\phi) \\ &\pm\frac{2\pi|n|}{2\sqrt{\det Q}}\int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x})\,\log(\chi\pm e^{-\phi})\star 1\,. \end{aligned} \tag{4.17}$$

We observe that the bulk part of (4.17) coincides with the bulk part of the action (3.16) while the boundary and source terms of (4.17) and (3.16) differ. This is in agreement with the remark made in section 2 regarding field redefinitions and Q-branes: there exists no field redefinition that relates IIB supergravity coupled to a Q-brane to IIB supergravity coupled to a D-brane.

Consider the equations of motion of χ and ϕ that follow from (4.17). These are

$$\partial_m(e^{2\phi}\partial^m\chi) = \mp\frac{\pi|n|}{\sqrt{\det Q}\,(e^{-\phi}\pm\chi)}\delta^{(10)}(\vec{x}), \tag{4.18}$$

$$\partial_m\partial^m\,\phi + e^{2\phi}\,(\partial\chi)^2 = -\frac{\pi|n|\,e^{-\phi}}{\sqrt{\det Q}\,(e^{-\phi}\pm\chi)}\delta^{(10)}(\vec{x}). \tag{4.19}$$

Using the Bogomol'nyi bound (3.5) which is related to (4.3) via eqs. (4.16) one reduces eqs. (4.18) and (4.19) to

$$\partial_m(e^{2\phi}\partial^m\chi) = \mp\frac{\pi|n|}{\sqrt{\det Q}(e^{-\phi}\pm\chi)}\delta^{(10)}(\vec{x}) \quad\rightarrow\quad \partial_m\partial^m(e^{\phi}) = -\frac{\pi|n|}{\sqrt{\det Q}(e^{-\phi}\pm\chi)}\delta^{(10)}(\vec{x}). \tag{4.20}$$

[7] The field redefinition (4.16) does not follow from the Lorentzian IIB field redefinitions (2.10) and (2.11) via Wick rotation. Here we consider the way in which the D- and Q-instantons differ from the point of view of classical Euclidean field theory.

Comparing (3.10) with (4.20) we see that the difference is in the factor $\frac{1}{2\sqrt{\det Q}\,(e^{-\phi}\pm\chi)}$ in the source term of the latter.

The Ansatz (3.5) and eqs. (4.20) must be compatible with the boundary condition (4.8) required by the shift symmetry of the axion χ'. Indeed, in terms of ϕ and χ which satisfy (3.5), eq. (4.8) takes the form

$$\int_{\partial\mathcal{M}_{10}} 2\sqrt{\det Q}\,(e^{-\phi}\pm\chi)\,e^{2\phi}\,\star\,d\chi = \mp\,2\pi|n|\int_{\mathcal{M}_{10}}\delta^{(10)}(\vec{x})\star 1\,, \tag{4.21}$$

which is consistent with (4.18) and (4.20).

Finally, in terms of the boundary values of ϕ and χ the on-shell action (4.13) for the Q-instanton has the form

$$S_Q\,|_{\text{on-shell}}= \frac{\pi|n|}{2\sqrt{\det \mathrm{Q}}}\,\log\frac{\chi_\infty + e^{-\phi_\infty}}{\chi_\infty - e^{-\phi_\infty}}\,, \tag{4.22}$$

which obviously differs from the D-instanton on-shell action (3.17). We conclude that from the point of view of the classical Euclidean field theory the difference between the D- and the Q-instanton lies in the different source and boundary terms.

In (4.22) χ_∞ and ϕ_∞ appear in the combinations $\chi_\infty + e^{-\phi_\infty}$ and $\chi_\infty - e^{-\phi_\infty}$, that are naturally associated to the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$, and not to the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$. The Euclidean path integral is invariant under $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ nonlinear transformations (see section 5). It is therefore not possible to obtain (4.22) with χ_∞ and ϕ_∞ denoting the RR axion and the dilaton, respectively, from a path integral analysis. This shows from a somewhat different point of view that the Q-instanton differs from the D-instanton.

As will be shown in section 5 the starting point of the path integral analysis is the first order action (2.19). From the path integral perspective the distinction between a Q- and a D-instanton is that there does not exist a local field redefinition that relates the action (2.19) for $\det Q = 0$ to the action (2.19) for $\det Q > 0$. The reason being that (2.19) depends both on $\mathcal{F}_9$ and χ'.

4.2 Q-instanton charge quantization

The quantization of the Q-instanton charge, eq. (4.8), follows from the standard Dirac–Nepomechie-Teitelboim quantization condition [14 – 16] applied to the Q(-1)-brane (instanton) and a Euclidean Q7-brane in a way similar to the D-instanton case [5]. Assume that the spatial volume of the 7-brane is compact with the topology of S^7. If we keep one point on the S^7 surface fixed and transport the 7-brane along closed paths its world-volume will have the topology of S^8. The wave function of this compact 7-brane will acquire, due to its minimal coupling to the axion dual 8-form (2.18) (with $m = 1$), the following phase factor

$$e^{i\int_\Sigma \mathcal{A}_8}\,, \tag{4.23}$$

where Σ is the world-volume of the compact 7-brane. Using Stokes' theorem we can write

$$\int_\Sigma \mathcal{A}_8 = \int_S \mathcal{F}_9 = -\int_{S'} \mathcal{F}_9\,, \tag{4.24}$$

where S and S' are the two capping surfaces of the world-volume $\Sigma = S^8$. The single-valuedness of the wave function (4.23) requires that

$$\int_{S^9} \mathcal{F}_9 = 2\pi n \,, \tag{4.25}$$

where $S^9 = S \bigcup S'$. Taking now into account the duality relation between $\mathcal{F}_9$ and the Wick rotated axion χ', as follows by varying eq. (4.1) with respect to $\mathcal{F}_9$, we arrive at eq. (4.8) which relates the value of the Q-instanton boundary term to its *quantized* charge.

4.3 The half-BPS condition

We will show that the Bogomol'nyi bound (4.3) also follows by analyzing the Killing spinor equations. In the Lorentzian IIB theory with vanishing 3- and 5-form field strengths the Killing spinor equations are

$$\delta\Psi_m = \left(\nabla_m - \frac{i}{2}Q_m\right)\epsilon \,, \tag{4.26}$$

$$\delta\lambda = iP_m\gamma^m\epsilon_C \,, \tag{4.27}$$

where $\epsilon = \epsilon_1 + i\epsilon_2$ and $\epsilon_C = \epsilon_1 - i\epsilon_2$ with ϵ_1 and ϵ_2 being Majorana-Weyl spinors. The coset Zweibein P_m and the composite $\mathrm{U}(1)$ connection Q_m of the axion-dilaton coset space $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \sim \frac{\mathrm{SU}(1,1)}{\mathrm{SU}(1)}$ must satisfy the Bianchi identity

$$dP - 2iQ \wedge P = 0 \,, \tag{4.28}$$

where P_m is such that the action (2.14) can be written as follows (see e.g. [17, 2] for details)

$$S = \int_{\mathcal{M}_{9,1}} \left(\star 1R - 2\star P \wedge \bar{P}\right) . \tag{4.29}$$

We choose a $\mathrm{U}(1)$ gauge in which

$$P_m = \frac{1}{2}\frac{\partial_m T}{(T^2 - 4\det Q)^{1/2}} + \frac{i}{2}(T^2 - 4\det Q)^{1/2}\partial_m\chi' \,, \tag{4.30}$$

$$Q_m = \frac{T}{2}\partial_m\chi' \,. \tag{4.31}$$

We Wick rotate eqs. (4.26) and (4.27) by sending χ' to $i\chi'$. Treating eqs. (4.26) and (4.27) and their complex conjugates separately we obtain

$$\left(\nabla_m + \frac{T}{4}\partial_m\chi'\right)\epsilon = 0 \,, \tag{4.32}$$

$$\left(\nabla_m - \frac{T}{4}\partial_m\chi'\right)\epsilon_C = 0 \,, \tag{4.33}$$

$$\left(\frac{\partial_m T}{(T^2 - 4\det Q)^{1/2}} - (T^2 - 4\det Q)^{1/2}\partial_m\chi'\right)\gamma^m\epsilon_C = 0 \,, \tag{4.34}$$

$$\left(\frac{\partial_m T}{(T^2 - 4\det Q)^{1/2}} + (T^2 - 4\det Q)^{1/2}\partial_m\chi'\right)\gamma^m\epsilon = 0 \,, \tag{4.35}$$

where ϵ and ϵ_C are Wick rotated spinors. The 1/2 BPS condition for the Q-instanton is

$$\partial_m \chi' = (T^2 - 4\det Q)^{-1} \partial_m T\,, \qquad \epsilon = 0\,, \tag{4.36}$$

and for the anti-Q-instanton

$$\partial_m \chi' = -(T^2 - 4\det Q)^{-1} \partial_m T\,, \qquad \epsilon_C = 0\,. \tag{4.37}$$

Since locally the Q-instanton and D-instanton solutions and the corresponding Killing spinor equations are related via a local field redefinition between the (T, χ') and (ϕ, χ) basis the Q-instanton solution solves the above Killing spinor equation. This has been explicitly verified. When either (4.36) or (4.37) holds we have that the Q- or anti-Q-instanton source term (4.6) is half BPS. Using $g_{mn} = \delta_{mn}$ it follows that for the anti-Q-instanton the Killing spinor ϵ is given by

$$\epsilon = (T^2 - 4\det Q)^{1/8} \epsilon_0\,, \tag{4.38}$$

where ϵ_0 is a constant spinor.

5. Path integral approach to Q-instantons

In this section we will justify the approach taken in section 4 by deriving the saddle point approximation of transition amplitudes between axion conjugate momentum eigenstates. Further, the imaginary part that, as we mentioned, should be added to the on-shell action, eq. (4.14), will be shown to follow from a Fourier transformation relating axion conjugate momentum eigenstates and axion field eigenstates. The discussions and arguments presented in this section are inspired by [18, 19]. We refer to [20 – 23] for related work in four dimensions.

5.1 Wick rotated path integrals and axions

In classical field theory when going from the Lorentzian IIB supergravity to Wick rotated Euclidean IIB supergravity we replace χ' by $i\chi'$. Here, we will show that on the level of the path integral χ' does not get replaced by $i\chi'$ when Wick rotating the path integral.

Consider the path integral

$$\int \mathcal{D}T\, \mathcal{D}\mathcal{F}_9\, \mathcal{D}\chi'\, e^{iS\,[T, \mathcal{F}_9, \chi']}\,, \tag{5.1}$$

with $S\,[T, \mathcal{F}_9, \chi']$ as given in (2.19). We do not include the metric in the discussion concerning the path integral since the metric for the instanton solutions is flat. The axion χ' in (5.1) can be integrated over using the identity

$$\int \mathcal{D}\chi'\, e^{-i\chi' d\mathcal{F}_9} = \delta[d\mathcal{F}_9]\,, \tag{5.2}$$

where $\delta[d\mathcal{F}_9]$ is a delta-functional, which implies that $d\mathcal{F}_9 = 0$.

The Wick rotated version of (5.1) is

$$\int \mathcal{D}T\, \mathcal{D}\mathcal{F}_9\, \mathcal{D}\chi'\, e^{-S_E\,[T, \mathcal{F}_9, \chi']}\,, \tag{5.3}$$

where $S_E = -iS$(Wick rotated) is given by (leaving out the metric)

$$S_E\left[T, \mathcal{F}_9, \chi'\right] = \int_{\mathcal{M}_{10}} \left(\frac{1}{2} \frac{1}{T^2 - 4\det Q} \star dT \wedge dT \right. \\ \left. + \frac{1}{2} \frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 + i\chi' d\mathcal{F}_9 \right) . \tag{5.4}$$

In the path integral (5.3) we are integrating over paths of field configurations with Dirichlet boundary conditions for the fields T and $\mathcal{F}_9$ while free or no boundary conditions are imposed for the field χ'. These boundary conditions are the same as those imposed on the variations of the action (2.19) with respect to T, $\mathcal{F}_9$ and χ'. The variation of χ' is entirely free without any boundary conditions because it appears in (5.4) without a derivative.

Notice that χ' in (5.4) has not been replaced by $i\chi'$. Now in the Euclidean path integral χ' can be again integrated out using the identity (5.2) which allows one to go to a second order formalism. If instead we had replaced χ' by $i\chi'$ this would have no longer been possible and the first order action in the Euclidean path integral would not have been equivalent to an 8-form gauge theory anymore since the Bianchi identity $d\mathcal{F}_9 = 0$ and its consequence $\mathcal{F}_9 = d\mathcal{A}_8$ would not arise.

5.2 The role of the moduli space

Let us rewrite the last term in (5.4) as follows

$$i\int_{\mathcal{M}_{10}} \chi' d\mathcal{F}_9 = -i\int_{\mathcal{M}_{10}} d\chi' \wedge \mathcal{F}_9 + i\int_{\partial\mathcal{M}_{10}} \chi' \mathcal{F}_9 \,. \tag{5.5}$$

If we require that the Euclidean path integral respects the standard IIB symmetry $\chi' \to \chi' + b$ where b is any real number then we find that $\mathcal{F}_9$ should satisfy the following boundary condition

$$b\int_{\partial\mathcal{M}_{10}} \mathcal{F}_9 = 2\pi n \qquad \text{with } n \in \mathbb{Z} \,. \tag{5.6}$$

Since b is arbitrary this means that $\int_{\partial\mathcal{M}_{10}} \mathcal{F}_9$ has to vanish. This would mean that there is no instanton present. If instead we only require that the axion can undergo integer, in particular, unit shifts $\chi' \to \chi' + 1$ then we find that

$$\int_{\partial\mathcal{M}_{10}} \mathcal{F}_9 = 2\pi n \qquad \text{with } n \in \mathbb{Z} \,. \tag{5.7}$$

We conclude from this that instantons can only exist in axion-dilaton theories whose moduli space is given by (2.2). We mention that the situation with the 7-brane solutions is in this respect entirely analogous. There the arguments to use (2.2) are based on the requirement of having 7-brane solutions with finite energy [24]. The conclusion that one must factor the moduli space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ by $\mathrm{SL}(2,\mathbb{Z})$ in order to even speak about instantons is clear from the path integral point of view and does not follow from the classical field theory approach of the previous two sections.

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5.3 Integrating over $\mathcal{F}_9$

Instead of integrating out χ' we shall now integrate (5.3) over $\mathcal{F}_9$. This is achieved by defining a new 9-form $\mathcal{F}_9'$

$$\mathcal{F}_9' = \mathcal{F}_9 + i(T^2 - 4\det Q) \star d\chi' \,. \tag{5.8}$$

Such a shift of $\mathcal{F}_9$ in the imaginary direction does not affect the integration in (5.3). The action (5.4) now becomes

$$\begin{aligned} S_E\left[T, \mathcal{F}_9', \chi'\right] = \int_{\mathcal{M}_{10}} \Bigg(& \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT + \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9' \wedge \mathcal{F}_9' \\ & + \frac{1}{2}(T^2 - 4\det Q) \star d\chi' \wedge d\chi' \Bigg) + i\int_{\partial\mathcal{M}_{10}} \chi'\mathcal{F}_9 \,. \end{aligned} \tag{5.9}$$

Even though $\mathcal{F}_9$ appears in the boundary term of (5.9) the $\mathcal{F}_9'$ integral is a Gaussian as we are integrating over $\mathcal{F}_9'$ with Dirichlet boundary conditions. The $\mathcal{F}_9$ in the boundary term is not integrated over, but is fixed by the identification $\chi' \sim \chi' + 1$, see eq. (5.7). Integrating over $\mathcal{F}_9'$ we find the following path integral

$$\int_F (T^2 - 4\det Q)^{1/2} \mathcal{D}T\, \mathcal{D}\chi'\, e^{-\tilde{S}_E\,[T,\chi']} \,, \tag{5.10}$$

where F below the integral sign means to indicate that we are only integrating over the paths of field configurations that are within the fundamental domain of the quantum moduli space (2.2). From now on this will always be assumed and the label F will be suppressed. The integration measure[8] now contains the factor $(T^2-4\det Q)^{1/2}$ and the Euclidean action $\tilde{S}_E\,[T,\chi']$ is given by

$$\begin{aligned} \tilde{S}_E\,[T,\chi'] = & \int_{\mathcal{M}_{10}} \left(\frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT + \frac{1}{2}(T^2 - 4\det Q) \star d\chi' \wedge d\chi' \right) \\ & + i\int_{\partial\mathcal{M}_{10}} \chi'\mathcal{F}_9 \,. \end{aligned} \tag{5.11}$$

5.4 Splitting the χ' integration into bulk and boundary integrations

We split up the integration over χ' into two pieces: the integration over bulk χ' field configurations and the integration over boundary χ'_∂ field configurations. The bulk χ' field configurations will be denoted by the same symbol as was used in the previous Subsections. Since we now explicitly write χ'_∂ for the boundary values this should cause no confusion. This split is most easily done using Dirichlet boundary conditions for the paths appearing in the path integral over the bulk χ' field configurations. If we do this then we can write for (5.10)

$$\int (T^2 - 4\det Q)^{1/2} \mathcal{D}T\, \mathcal{D}\chi' \mathcal{D}\chi'_\partial\, e^{-S_E\,[T,\chi',\chi'_\partial]} \,, \tag{5.12}$$

[8]In terms of τ and $\bar{\tau}$ the integration measure would be $(\operatorname{Im}\tau)^{-2}\mathcal{D}\tau\mathcal{D}\bar{\tau}$, which is $\mathrm{PSL}(2,\mathbb{Z})$ invariant.

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with Dirichlet boundary conditions on the integrations over T and χ'. The action appearing in (5.12) is given by

$$S_E\,[T,\chi',\chi'_\partial] = \int_{\mathcal{M}_{10}} \left(\frac{1}{2}\frac{1}{T^2-4\det Q} \star dT\wedge dT + \frac{1}{2}(T^2-4\det Q)\star d\chi'\wedge d\chi'\right) \\ + i\int_{\partial\mathcal{M}_{10}} \chi'_\partial\,\mathcal{F}_9\,. \tag{5.13}$$

The variation of the bulk part of (5.13) with respect to T and χ' satisfying Dirichlet boundary conditions produces the standard (non-Wick rotated) IIB axion-dilaton equations of motion.

5.5 Tunneling interpretation

In this Subsection we will discuss what is precisely computed by the Euclidean path integral (5.3), i.e. by (5.12).

We would like to interpret (5.12) in terms of matrix elements describing a tunneling process from an initial ($t=-\infty$) time-like hypersurface Σ_i to a final ($t=+\infty$) time-like hypersurface Σ_f. The time-like hypersurfaces Σ_i and Σ_f constitute surfaces on which field operator states exist. In order to describe this within the space-time $\mathcal{M}_{9,1}$ we add to it spatial infinity as a point. Hence we consider $\mathcal{M}_{9,1}\cup\{r=\infty\}$, where r is a radial coordinate. The topology of this one-point compactified space-time is given by $\mathbb{R}\times S^9$ whose boundary $\partial\,(\mathcal{M}_{9,1}\cup\{r=\infty\})$ is given by the disjoint union $\Sigma_i\cup\Sigma_f$ where the initial and final time-like hypersurfaces have the topology of S^9.

The instanton charge $\int_{\partial\mathcal{M}_{10}}\mathcal{F}_9$ that appears in the imaginary part of eq. (5.13) is equal to $\int_{\Sigma_f}\mathcal{F}_9^f - \int_{\Sigma_i}\mathcal{F}_9^i$. Multiplying this equality by χ'_∂ we can write

$$i\int_{\partial\mathcal{M}_{10}}\chi'_\partial\,\mathcal{F}_9 = i\int_{\Sigma_f}\chi'_\partial\,\mathcal{F}_9^f - i\int_{\Sigma_i}\chi'_\partial\,\mathcal{F}_9^i\,, \tag{5.14}$$

where the values of the axion χ' on the initial and final timelike hypersurfaces Σ_i and Σ_f are the same: $\chi'_i=\chi'_f=\chi'_\partial$. In the following we will write $\chi'_\partial=\chi'_\infty$. Further we have

$$\int\mathcal{D}\chi'_\partial e^{-i\int_{\partial\mathcal{M}_{10}}\chi'_\partial\mathcal{F}_9} = \int\mathcal{D}\chi'_i\mathcal{D}\chi'_f\delta(\chi'_i-\chi'_f)e^{-i\int_{\Sigma_f}\chi'_f\mathcal{F}_9^f+i\int_{\Sigma_i}\chi'_i\mathcal{F}_9^i}\,. \tag{5.15}$$

The boundary states in (5.12) at $\Sigma_{i,f}$ satisfy (5.7) and (5.14) and so the $\mathcal{F}_9^{i,f}$ boundary data are on-shell.

We will now show that one can use the duality relation (2.15) restricted to the surfaces $\Sigma_{i,f}$ to interpret the boundary data $\mathcal{F}_9^{i,f}$ of eq. (5.15) in terms of the axion momentum, or equivalently, in terms of the Noether charge density associated with the axion shift symmetry. Note that the time component of the Noether current (charge density) is equal to the axion χ' canonical momentum π' obtained by varying the Lagrangian (2.14) with respect to $\partial_0\chi'$

$$\pi' = \frac{\delta\mathcal{L}}{\delta(\partial_0\chi')} = (T^2-4\det Q)\,\partial_0\chi' = J_N^0\,. \tag{5.16}$$

Let us consider tunneling between canonical momentum eigenstates of the axion χ' (or equivalently between its Noether charge eigenstates) from the initial surface Σ_i to the final surface Σ_f. These are described by the following matrix element

$$\lim_{\Delta T\to\infty}\langle\pi'_f \mid e^{-H\Delta T} \mid \pi'_i\rangle\,, \tag{5.17}$$

where ΔT is the Wick rotated time interval between Σ_i and Σ_f, H is the axion-dilaton Hamiltonian which can be obtained from the (flat metric) action (2.14) by the Legendre transformation and $\pi'_{i,f}$ are the initial and final momenta of the axion.

The matrix element (5.17) is related by a Fourier transformation to the matrix element describing the transition between two boundary eigenstates χ'_i and χ'_f of the axion. Namely, (for $\Delta T\to\infty$) we have

$$\langle\pi'_f \mid e^{-H\Delta T} \mid \pi'_i\rangle = \int \mathcal{D}\chi'_i\mathcal{D}\chi'_f e^{-i\int_{\Sigma_f}\chi'_f\,\pi'_f + i\int_{\Sigma_i}\chi'_i\,\pi'_i}\langle\chi'_f \mid e^{-H\Delta T} \mid \chi'_i\rangle\,, \tag{5.18}$$

where

$$\langle\chi'_f \mid e^{-H\Delta T} \mid \chi'_i\rangle = \langle\chi'_f \mid e^{-H\Delta T} \mid \chi'_f\rangle\delta(\chi'_i-\chi'_f)\,. \tag{5.19}$$

We see that no tunneling takes place between vacua for which $\chi'_i\neq\chi'_f$. This means that the value of χ' at, say, $t=+\infty$ acts as a superselection parameter, like the theta parameter in Yang-Mills theory. Hence, physical processes in vacua with different values of χ'_f are not correlated.

The matrix element appearing on the right-hand side of eq. (5.19) is given by

$$\langle\chi'_f \mid e^{-H\Delta T} \mid \chi'_f\rangle = \int (T^2-4\det Q)^{1/2}\mathcal{D}T\,\mathcal{D}\chi'\,e^{-S_E\,[T,\chi']}\,, \tag{5.20}$$

with Dirichlet boundary conditions on the integrations over T and χ' and

$$S_E\,[T,\chi'] = \int_{\mathcal{M}_{10}}\left(\frac{1}{2}\frac{1}{T^2-4\det Q}\star dT\wedge dT + \frac{1}{2}(T^2-4\det Q)\star d\chi'\wedge d\chi'\right)\,. \tag{5.21}$$

We now compare eqs. (5.18)–(5.21) with (5.12)–(5.15). Eqs. (5.18)–(5.21) taken together provide a closed expression for the matrix element on the left hand-side of eq. (5.18). On the other hand eqs. (5.12)–(5.15) provide an expression for the path integral in (5.12). In order that (5.12) computes a physical quantity, namely the matrix element of (5.18), we choose the boundary values of $\mathcal{F}_9$ to be associated with the boundary values of the χ' canonical momentum (5.16) via the duality relation (2.15)

$$\int_{\Sigma_{i,f}}\mathcal{F}_9^{i,f} = \int_{\Sigma_{i,f}}\star(T^2-4\det Q)d\chi' = \int_{\Sigma_{i,f}} J^0_N\,d\Omega_9 = \int_{\Sigma_{i,f}}\pi'_{i,f}\,d\Omega_9\,, \tag{5.22}$$

where $d\Omega_9$ denotes the integration measure of the unit 9-sphere.

Using the inverse Fourier transform we have

$$\langle\chi'_f \mid e^{-H\Delta T} \mid \chi'_i\rangle = \int \mathcal{D}\pi'_i\mathcal{D}\pi'_f e^{i\int_{\Sigma_f}\chi'_f\,\pi'_f - i\int_{\Sigma_i}\chi'_i\,\pi'_i}\langle\pi'_f \mid e^{-H\Delta T} \mid \pi'_i\rangle\,, \tag{5.23}$$

where now

$$\langle \pi'_f \mid e^{-H\Delta T} \mid \pi'_i \rangle = \int \mathcal{D}T\, \mathcal{D}\mathcal{F}_9\, \delta[d\mathcal{F}_9]\, e^{-S_E\,[T,\mathcal{F}_9]}\,, \tag{5.24}$$

with Dirichlet boundary conditions imposed on the integrations over T and $\mathcal{F}_9$ and where

$$S_E\,[T,\mathcal{F}_9] = \int_{\mathcal{M}_{10}} \left(\frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT + \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 \right)\,. \tag{5.25}$$

5.6 Saddle point approximation

The saddle point approximation of $\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle$ can be obtained using eqs. (5.23), (5.24) and (5.25). For a single instanton of charge n we have

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle \simeq N e^{i\int_{\Sigma_f} \chi'_f\, \pi'_f - i\int_{\Sigma_i} \chi'_f\, \pi'_i} e^{-S_E\,[T,\mathcal{F}_9]}\,|_{\text{on-shell}}\,, \tag{5.26}$$

where N is a prefactor that we will not attempt to evaluate. On the mass shell we have

$$\int_{\Sigma_f} \chi'_f\, \pi'_f d\Omega_9 - \int_{\Sigma_i} \chi'_f\, \pi'_i d\Omega_9 = 2\pi n \chi'_\infty\,, \tag{5.27}$$

which follows from eqs. (5.7) and (5.14). Further, on-shell and outside the Q-instanton source $S_E\,[T,\mathcal{F}_9] = S_E\,[T, d\mathcal{A}_8]$. The on-shell action can be written as the sum of a quadratic term and a rest term as

$$S_E\,[T, d\mathcal{A}_8] = \frac{1}{2}\int_{\mathcal{M}_{10}} \frac{1}{T^2 - 4\det Q} \star (dT \mp \star\mathcal{F}_9) \wedge\, (dT \mp \star\mathcal{F}_9) \pm G\,, \tag{5.28}$$

with G given by

$$G = \int_{\mathcal{M}_{10}} \frac{1}{T^2 - 4\det Q} dT \wedge \mathcal{F}_9 = -\int_{\partial\mathcal{M}_{10}} \frac{1}{4\sqrt{\det Q}} \log\left(\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \right) \mathcal{F}_9\,, \tag{5.29}$$

where $\partial\mathcal{M}_{10} = \partial\mathcal{M}_\infty + \partial\mathcal{M}_0$. The boundaries $\partial\mathcal{M}_\infty$ and $\partial\mathcal{M}_0$ are, respectively, the 9-sphere at infinity and around the origin where the field strength F_9 fails to be exact (the location of its magnetic source). However because at $|\vec{x}| = 0$ the field T blows up, the value of G is zero at this point and only the boundary at infinity contributes. The first term in the action (5.28) is positive definite. We thus have the following Bogomol'nyi bound for field configurations respecting the symmetries of the Q(-1)-brane solution

$$S_I \geq \pm G\,. \tag{5.30}$$

Solutions that satisfy the Bogomol'nyi bound must have the property that

$$dT = \pm \star \mathcal{F}_9\,. \tag{5.31}$$

For such configurations the on-shell value of the action is given by

$$S_E\,[T, d\mathcal{A}_8]\,|_{\text{on-shell}} = -G = \frac{\pi|n|}{2\sqrt{\det Q}} \log\left(\frac{T_\infty + 2\sqrt{\det Q}}{T_\infty - 2\sqrt{\det Q}} \right)\,, \tag{5.32}$$

where $T_\infty > 2\sqrt{\det Q}$ is the asymptotic value of T. The result (5.32) agrees with (4.13) and provides a saddle point approximation of the matrix element of a transition between axion charge eigenstates (or conjugate momentum eigenstates).

Using eqs. (5.27) and (5.32) we find for the saddle point approximation (5.26),

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle \simeq N e^{2\pi n i \chi'_\infty - 2\pi |n| \mathrm{Im}\mathcal{T}_\infty} = \begin{cases} N e^{2\pi n i \mathcal{T}_\infty} & \text{for } n > 0\,, \\ N e^{2\pi n i \bar{\mathcal{T}}_\infty} & \text{for } n < 0\,. \end{cases} \tag{5.33}$$

The case $n > 0$ corresponds to the Q-instanton whereas $n < 0$ corresponds to the anti-Q-instanton. We thus see that adding the term (4.14) to the action (4.13) leads to a saddle point approximation of the matrix element of the transition between axion eigenstates $\chi'_i = \chi'_f = \chi'_\infty$. The result (5.33) will be used in the next section to argue that the $\mathcal{R}^4$ terms near the points i and ρ of figure 1 receive contributions from Q-instantons.

6. Q-instanton contributions to the $\mathcal{R}^4$ terms

The $\mathcal{R}^4$ terms are those terms in the effective action that are of order $(\alpha')^3$ relative to the Einstein-Hilbert term. In [6] it is argued that the part of the $\mathcal{R}^4$ terms that only contains derivatives of the metric is multiplied by a $\mathrm{PSL}(2,\mathbb{Z})$ invariant real-analytic modular form, a generalized Eisenstein series. Such functions are eigenfunctions of the Laplace operator on the hyperbolic plane. In [25] it is shown that this picture is confirmed by requiring supersymmetry at the order $(\alpha')^3$ relative to the Einstein-Hilbert term. The $\mathcal{R}^4$ terms contain besides derivatives of the metric also contributions involving terms with derivatives of the other bosonic fields of the type IIB theory. For the NSNS fields and the RR 0-form a conjectured $\mathrm{SL}(2,\mathbb{Z})$ invariant $\mathcal{R}^4$ term is proposed in [26]. Here we will only consider the part of the $\mathcal{R}^4$ terms that involves derivatives of the metric and that can be obtained by considering on-shell amplitudes for four graviton scattering. We write [26]

$$\mathcal{R}^4 = f(\tau,\bar\tau)\left(t_8^{abcdefgh} t_8^{mnpqrstu} + \frac{1}{8}\epsilon_{10}^{abcdefghij}\epsilon_{10}{}^{mnpqrstu}{}_{ij}\right) R_{abmn}R_{cdpq}R_{efrs}R_{ghtu} + \cdots\,, \tag{6.1}$$

where t_8 is defined in [27], ϵ_{10} is the 10-dimensional Levi-Cività tensor and R_{abmn} is the Riemann tensor. The dots indicate that there are more contributions to $\mathcal{R}^4$.

The function $f(\tau,\bar\tau)$, a generalized Eisenstein series, has the form

$$f(\tau,\bar\tau) = \sum_{(p,n)\neq(0,0)} \frac{\tau_2^{3/2}}{|p+n\tau|^3}\,, \tag{6.2}$$

where $\tau = \tau_1 + i\tau_2$ and the sum is over all integers $p, n \in \mathbb{Z}$ except when both p and n are zero. In order to see the contributions coming from single multiply-charged D- and anti-D-instantons one writes f as a Fourier series in $\tau_1 = \chi$. We have [28]

$$\begin{aligned} f(\tau,\bar\tau) &= 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3}\tau_2^{-1/2} + 8\pi\tau_2^{1/2}\sum_{m\neq 0}\sum_{n=1}^{\infty}|\frac{m}{n}|e^{2\pi i m n\tau_1}\, K_1(2\pi|mn|\tau_2) \\ &= 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3}\tau_2^{-1/2} + 8\pi\tau_2^{1/2}\sum_{k=1}^{\infty} k\,\sigma_{-2}(k)\left(e^{2\pi i k\tau_1} + e^{-2\pi i k\tau_1}\right)\, K_1(2\pi k\tau_2)\,, \end{aligned} \tag{6.3}$$

with $\sigma_{-2}(k)$ given by

$$\sigma_{-2}(k) = \sum_{d|k} d^{-2}\,, \tag{6.4}$$

where the sum is over all positive divisors d of k. The expression (6.3) is a cosine series with coefficients $16\pi\tau_2^{1/2}\,k\,\sigma_{-2}(k)\,K_1(2\pi k\tau_2)$, where K_1 is the modified Bessel function of the second kind. The τ_1 independent terms in (6.3) do not come from D-instantons, instead they come from an $(\alpha')^3$ tree level and a one-loop effect in the four graviton amplitude [6].

In order to see the contribution from single multiply-charged D- and anti-D-instantons one considers (6.3) close to $\tau_0 = i\infty$, i.e. in the limit $\tau_2 \to \infty$. Using that for $x \to \infty$ we have $K_1(x) = \sqrt{\frac{\pi}{2x}}e^{-x}(1+\cdots)$ we find that at the leading order in the limit $\tau_2 \to \infty$

$$f(\tau,\bar\tau) \approx 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3}\tau_2^{-1/2} + 4\pi\sum_{k=1}^{\infty} k^{1/2}\,\sigma_{-2}(k)\left(e^{2\pi i k\tau} + e^{-2\pi i k\bar\tau}\right)\,. \tag{6.5}$$

In the exponents of (6.5) one recognizes the D-instanton action (3.19).

Since the Q-instantons preserve precisely the same supersymmetries as the D-instanton (see Subsection 4.3) they also contribute to the function $f(\tau,\bar\tau)$. To justify this argument, in the remainder of this section we shall Fourier expand the function f in terms of χ' and compute the Fourier coefficients which will be functions of T. This will result in an exact expression for f which is analogous to eq. (6.3). Schematically we write

$$f(T,\chi') = \sum_{n=-\infty}^{\infty} c_n(T)e^{2\pi i n\chi'}\,, \tag{6.6}$$

where $c_n(T)$ are the Fourier coefficients. This series is manifestly invariant under $\chi' \to \chi'+1$ and provides us with the behavior of f near $\tau = \tau_0$. It will be shown that f consists of a χ' independent part and of a cosine series that corresponds to an infinite sum of single multiply-charged Q- and anti-Q-instantons.

Since the Q-instantons are defined near the points $\tau = i, \rho$ in the moduli space they do not contribute to the low energy effective action in regions of moduli space where perturbative string theory is valid, i.e. around $\tau_0 = i\infty$. We shall expand the function f around the fixed points $\tau_0 = i, \rho$ of the axion-dilaton moduli space. To this end it will prove convenient to introduce what we refer to as the (η,φ) coordinate system which is defined by the relation

$$\frac{\tau-\tau_0}{\tau-\bar\tau_0} = e^{i\varphi}\tanh\frac{\eta}{2}\,, \tag{6.7}$$

where φ and η are related to T and χ' (see eqs. (2.10) and (2.11)) as follows[9]

$$\varphi = 2\sqrt{\det Q}\,\chi' \qquad \text{where} \qquad 0 \le \varphi < 2\pi\,, \tag{6.9}$$

[9]The transformation from T to η is such that the kinetic term for T in the action becomes canonically normalized, i.e. we have

$$\frac{\star dT\wedge dT}{T^2 - 4\det Q} = \star d\eta\wedge d\eta\,. \tag{6.8}$$

$$\tanh\frac{\eta}{2} = e^{-2\sqrt{\det Q}\,\mathrm{Im}\,T} \qquad \text{where} \qquad 0 < \eta < \infty\,. \tag{6.10}$$

Substituting (6.7) into (6.2) we obtain

$$f(\eta,\varphi) = \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\tau_0)^{3/2}}{|p+n\tau_0|^3}\frac{1}{(\cosh\eta+\sinh\eta\cos(\varphi+\beta(p,n;\tau_0)))^{3/2}}\,, \tag{6.11}$$

where $\beta(p,n;\tau_0)$ is defined by

$$\cos\beta(p,n;\tau_0) = \frac{n^2(\mathrm{Im}\,\tau_0)^2-(p+n\mathrm{Re}\,\tau_0)^2}{n^2(\mathrm{Im}\,\tau_0)^2+(p+n\mathrm{Re}\,\tau_0)^2}\,, \tag{6.12}$$

$$\sin\beta(p,n;\tau_0) = \frac{2n\,\mathrm{Im}\,\tau_0\,(p+n\mathrm{Re}\,\tau_0)}{n^2(\mathrm{Im}\,\tau_0)^2+(p+n\mathrm{Re}\,\tau_0)^2}\,. \tag{6.13}$$

From the definition of φ in terms of χ' it follows that the invariance of $f(T,\chi')$ under $\chi'\to\chi'+1$ implies the invariance of $f(\eta,\varphi)$ under $\varphi\to\varphi+2\sqrt{\det Q}$. Hence, we make the following Fourier series decomposition of $f(\eta,\varphi)$

$$f(\eta,\varphi) = \sum_{m=-\infty}^{\infty} a_{\frac{\pi m}{\sqrt{\det Q}}}(\eta)\, e^{\frac{\pi}{\sqrt{\det Q}}mi\varphi}\,. \tag{6.14}$$

The Fourier coefficients $a_{\frac{\pi m}{\sqrt{\det Q}}}$ are given by

$$a_{\frac{\pi m}{\sqrt{\det Q}}}(\eta) = \frac{1}{2\sqrt{\det Q}}\int_0^{2\sqrt{\det Q}} d\varphi\, f(\eta,\varphi)e^{-\frac{\pi}{\sqrt{\det Q}}mi\varphi}\,. \tag{6.15}$$

By using (6.11) and by shifting the integration over φ in (6.15) to an integration over $\theta=\varphi+\beta(p,n;\tau_0)$ the Fourier coefficients $a_{\frac{\pi m}{\sqrt{\det Q}}}$ can be written as

$$\begin{aligned} a_{\frac{\pi m}{\sqrt{\det Q}}}(\eta) =& \frac{1}{2\sqrt{\det Q}} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\tau_0)^{3/2}}{|p+n\tau_0|^3}\, e^{\frac{\pi}{\sqrt{\det Q}}mi\beta(p,n;\tau_0)}\times \\ &\times \int_{\beta(p,n;\tau_0)}^{2\sqrt{\det Q}+\beta(p,n;\tau_0)} d\theta \frac{e^{-\frac{\pi}{\sqrt{\det Q}}mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}}\,. \end{aligned} \tag{6.16}$$

We will further evaluate (6.16) for the cases $\tau_0=i$ and $\tau_0=\rho$ separately. We start with the case $\tau_0=i$. In table 2 we presented some data regarding the orbifold points $\tau_0=i,\rho$. We found that for $\tau_0=i$ we have $2\sqrt{\det Q}=\pi$. From eqs. (6.12) and (6.13) specified to the case $\tau_0=i$ we derive the following two identities

$$\beta(-n,p;i) = \pi+\beta(p,n;i)\,, \tag{6.17}$$

$$\beta(n,p;i) = \pi-\beta(p,n;i)\,. \tag{6.18}$$

Using the identity (6.17) we can write

$$\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \int_{\beta(p,n;i)}^{\pi+\beta(p,n;i)} d\theta \frac{e^{-2mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}} =$$
$$\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \int_{\pi+\beta(p,n;i)}^{2\pi+\beta(p,n;i)} d\theta \frac{e^{-2mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}} = \tag{6.19}$$
$$\frac{1}{2}\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \int_0^{2\pi} d\theta \frac{e^{-2mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}},$$

where in the last equality we took the average of the first two lines and used the property $\int_\beta^{2\pi+\beta} = \int_\beta^0 + \int_0^{2\pi} + \int_{2\pi}^{2\pi+\beta} = \int_0^{2\pi}$ because the integrand is periodic with the period 2π. We thus find that the Fourier coefficients (6.16) take the form

$$a_{2m}(\eta) = \frac{1}{2\pi}\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \int_0^{2\pi} d\theta \frac{\cos(2m\theta)}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}}, \tag{6.20}$$

where the integral from 0 to 2π that involves $\sin(2m\theta)$ vanished.

The identity (6.18) can be used to show that the sum preceding the integral in (6.20) satisfies

$$\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} = \sum_{(p,n)\neq(0,0)} \frac{e^{-2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}}, \tag{6.21}$$

so that $a_{2m}(\eta) = a_{-2m}(\eta)$. The latter property implies that the Fourier expansion (6.14) becomes the following cosine series

$$f(\eta,\varphi) = a_0(\eta) + \sum_{m=1}^{\infty} a_{2m}(\eta)\left(e^{2mi\varphi}+e^{-2mi\varphi}\right). \tag{6.22}$$

The integral in (6.20) is the integral representation (up to a factor) of a toroidal function, denoted by $P^{2m}_{1/2}(\cosh\eta)$. Toroidal or ring functions are special cases of the associated Legendre functions. We have [29]

$$\int_0^{2\pi} d\theta \frac{\cos(n\theta)}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}} = 2\pi(-1)^n \frac{\Gamma(\frac{3}{2}-n)}{\Gamma(\frac{3}{2})} P^n_{1/2}(\cosh\eta). \tag{6.23}$$

The functions $P^n_{1/2}(\cosh\eta)$ for $n=0,1,2,\ldots$ can be written in terms of a hypergeometric function as follows [29]

$$P^n_{1/2}(\cosh\eta) = \frac{1}{2^n}\frac{\Gamma(\frac{3}{2}+n)}{\Gamma(n+1)\Gamma(\frac{3}{2}-n)} \sinh^n\eta\, F\left(\frac{n}{2}+\frac{3}{4},\frac{n}{2}-\frac{1}{4};n+1;-\sinh^2\eta\right). \tag{6.24}$$

Substituting eqs. (6.23) and (6.24) into (6.20) we see that the function $f(\eta,\varphi)$, eq. (6.22), around the point $\tau_0 = i$ can be written as the following Fourier series

$$f(\eta,\varphi) = \sum_{(p,n)\neq(0,0)} \frac{1}{(p^2+n^2)^{3/2}} F\left(\frac{3}{4},-\frac{1}{4};1;-\sinh^2\eta\right) + \sum_{m=1}^{\infty}\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \times \tag{6.25}$$
$$\times \frac{1}{2^{2m}}\frac{\Gamma(\frac{3}{2}+2m)}{\Gamma(2m+1)\Gamma(\frac{3}{2})} \sinh^{2m}\eta\, F\left(m+\frac{3}{4},m-\frac{1}{4};2m+1;-\sinh^2\eta\right)\left(e^{2mi\varphi}+e^{-2mi\varphi}\right).$$

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From eqs. (6.9) and (6.10) we know that

$$\varphi = \pi\chi' \qquad \text{and} \qquad \sinh^2\eta = \frac{T^2 - 4\det Q}{4\det Q} \qquad \text{with} \qquad \sqrt{\det Q} = \frac{\pi}{2}\,. \tag{6.26}$$

The Fourier series (6.25) in terms of χ' and $T^2 - 4\det Q$ associated with the fixed point $\tau_0 = i$ is analogous to the Fourier series expansion (6.3) around the point $\tau_0 = i\infty$ in terms of $\tau_1 = \chi$ and $\tau_2 = \mathrm{Im}\,\tau = e^{-\phi}$.

In order to make manifest the Q- and anti-Q-instanton contributions to the function f we consider the expansion (6.25) at the leading order around the point $\eta = 0$ (that corresponds to a singular point of the associated Legendre function $P^{2m}_{1/2}(\cosh\eta)$). Note that, by virtue of the relation (6.7), the point $\eta = 0$ corresponds to $\tau = i$. Using that at leading order

$$\sinh^2\eta \approx 4\,e^{-2\pi \mathrm{Im}\,\mathcal{T}}\,, \tag{6.27}$$

we find that at this order[10]

$$\begin{aligned} f(\mathcal{T},\bar{\mathcal{T}}) \approx & \sum_{(p,n)\neq(0,0)} \frac{1}{(p^2+n^2)^{3/2}} \\ & + \sum_{m=1}^{\infty} \sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \frac{\Gamma(\frac{3}{2}+2m)}{\Gamma(2m+1)\Gamma(\frac{3}{2})} \left(e^{2\pi m i\mathcal{T}} + e^{-2\pi m i\bar{\mathcal{T}}}\right)\,. \end{aligned} \tag{6.29}$$

The form of the sum over $m = 1, 2, \ldots$ in eq. (6.29) which is analogous to the D-instanton case prompts us to assume that it reproduces the contribution of single multiply-charged Q- and anti-Q-instantons as one can see by comparing (6.29) with eq. (5.33). The first term in (6.29) does not correspond to an instanton contribution. Its origin is yet to be understood.

We have discussed in detail how to obtain the Fourier series expansion of the function f around $\tau_0 = i$, eq. (6.25). We end this section by briefly discussing the Fourier series expansion of f around $\tau_0 = \rho$. The starting point is eq. (6.16) in which we take $\tau_0 = \rho$ and $\sqrt{\det Q} = \frac{\pi}{3}$ see table 2. In this case from eqs. (6.12) and (6.13) we can obtain the following three identities

$$\begin{aligned} \beta(n, n-p;\rho) &= \frac{2\pi}{3} + \beta(p,n;\rho)\,, & (6.30)\\ \beta(p-n, p;\rho) &= \frac{4\pi}{3} + \beta(p,n;\rho) & (6.31)\\ -\beta(n,p;\rho) &= \frac{4\pi}{3} + \beta(p,n;\rho)\,. & (6.32) \end{aligned}$$

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[10]This can alternatively be derived by using that $P^n_{1/2}(\cosh\eta)$ can also be written as

$$P^n_{1/2}(\cosh\eta) = \frac{\Gamma(\frac{3}{2}+n)}{\Gamma(n+1)\Gamma(\frac{3}{2}-n)} \tanh^n\frac{\eta}{2}\, F\left(-\frac{1}{2},\frac{3}{2};1+n;-\sinh^2\frac{\eta}{2}\right)\,. \tag{6.28}$$

Then using that for $\tau_0 = i$ we have $\tanh^2\frac{\eta}{2} = e^{-2\pi \mathrm{Im}\,\mathcal{T}}$ and $n = 2m$ the result eq. (6.29) follows.

Using (6.30) and (6.31) one can show, in a way which is very similar to the derivation of eq. (6.20) for $\tau_0 = i$, that the Fourier coefficients $a_{3m}(\eta)$ are given by

$$a_{3m}(\eta) = \frac{1}{2\pi} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} e^{3mi\beta(p,n;\rho)} \int_0^{2\pi} d\theta \frac{\cos(3m\theta)}{(\cosh\eta + \sinh\eta\cos\theta)^{3/2}} \,. \tag{6.33}$$

It follows by employing eq. (6.32) that $a_{3m}(\eta) = a_{-3m}(\eta)$. Hence, using the Fourier decomposition (6.14) and eqs. (6.33), (6.23) and (6.24) we find for $\tau_0 = \rho$

$$\begin{aligned} f(\eta,\varphi) = &\sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} F\left(\frac{3}{4},-\frac{1}{4};1;-\sinh^2\eta\right) + \sum_{m=1}^{\infty} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} e^{3mi\beta(p,n;\rho)} \times \\ &\times (-1)^m \frac{1}{2^{3m}} \frac{\Gamma(\frac{3}{2}+3m)}{\Gamma(3m+1)\Gamma(\frac{3}{2})} \sinh^{3m}\eta F\left(\frac{3m}{2}+\frac{3}{4},\frac{3m}{2}-\frac{1}{4};3m+1;-\sinh^2\eta\right)\left(e^{3mi\varphi}+e^{-3mi\varphi}\right). \end{aligned} \tag{6.34}$$

At leading order we can write

$$\sinh^3\eta \approx 8\, e^{-2\pi \mathrm{Im}\,\mathcal{T}} \,, \tag{6.35}$$

so that at this order near $\tau_0 = \rho$ we obtain

$$\begin{aligned} f(\mathcal{T},\bar{\mathcal{T}}) \approx &\sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} + \sum_{m=1}^{\infty} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} e^{3mi\beta(p,n;\rho)} \times \\ &\times (-1)^m \frac{\Gamma(\frac{3}{2}+3m)}{\Gamma(3m+1)\Gamma(\frac{3}{2})} \left(e^{2\pi mi\mathcal{T}} + e^{-2\pi mi\bar{\mathcal{T}}}\right) \,, \end{aligned} \tag{6.36}$$

where we used $\varphi = \frac{2\pi}{3}\chi'$.

The expressions (6.29) for $\tau_0 = i$ and (6.36) for $\tau_0 = \rho$ can be contrasted with the leading order result for $\tau_0 = i\infty$, eq. (6.5). The results (6.29) and (6.36) differ from (6.5) most notably in the axion-independent parts. We expect that there to be a Q-brane interpretation for the χ' independent pieces of (6.29) and (6.36), but at this moment it is not clear what kind of processes would account for these terms.

7. Discussion

In this article we have constructed new 1/2 BPS instanton solutions to the Wick rotated Euclidean IIB supergravity theory. We have shown that they differ from the known D-instantons and that they are the electric partners of the Q7-branes of [1, 2]. The path integral approach to the Q-instantons shows the existence of new vacua and a new superselection parameter χ'_∞. Further, we have argued that the Q-instantons contribute to the $\mathcal{R}^4$ terms near the points $\tau_0 = i, \rho$ of the quantum moduli space $\frac{\mathrm{PSL}(2,\mathbb{R})}{\mathrm{SO}(2)\times\mathrm{PSL}(2,\mathbb{Z})}$. The expansion of the generalized Eisenstein series around the points $\tau_0 = i, \rho$ contains terms that do not depend on χ' and for which a Q-brane interpretation is yet to be found.

We believe that the results of this article together with [1, 2] support the idea that IIB supergravity provides a valid field theory approximation of some underlying quantum theory near each of the orbifold points of the quantum axion-dilaton moduli space of figure 1. It is of interest to understand if in addition to the Q7-branes and the Q-instantons there exist other Qp-brane solutions associated to the orbifold points i and ρ of the IIB quantum moduli space.

In addition to the instantons and 7-branes it should also be possible to consider 3-branes near the orbifold points $\tau_0 = i, \rho$ since the 3-brane is an $\mathrm{SL}(2,\mathbb{Z})$ singlet and can be put at any point of the IIB moduli space. Based on the arguments presented in this paper we expect there to exist a field theory description of the world-volume theory of a "Q3-brane", i.e. a 3-brane near $\tau_0 = i, \rho$. It would then be interesting to study such a Q3-brane in the presence of probe Q-instantons or probe Q7-branes and to see if one may learn something about the Yang-Mills theory on the Q3-branes.

We end this discussion section with the following comment on the relevance of Q-branes in relation to gauged supergravities. The idea that the type IIB supergravity theory can be used as a valid approximation of some underlying quantum theory near the points $\tau = i, \rho$ is of importance, for example, if one considers gauged supergravities that result from the IIB theory in which the $\chi' \to \chi' + b$ isometry has been gauged. The simplest example of such a gauged supergravity is the nine-dimensional SO(2) gauged maximal supergravity that is constructed via a Scherk-Schwarz reduction of IIB supergravity by gauging the SO(2) subgroup of $\mathrm{SL}(2,\mathbb{R})$ [30]. This nine-dimensional theory has domain-wall solutions which correspond (via uplifting) to Q7-branes of ten-dimensional IIB supergravity.[11] Thus, the study of the structure of gauged supergravities may provide us with additional information about the nature of Q-branes and whether they manifest yet unexplored corners of M-theory.

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